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Exercises: Charlotte Amann-Bartnick

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Winter term 2025/2026

Model Theory
Sheet 0
Not to be handed in
This sheet will be discussed during the first exercise class.

Exercise 1.

a) Let $\mathcal{L}$ be a language and $\varphi[x_1, \ldots, x_n]$ be an $\mathcal{L}$-formula of the form

$$\varphi[x_1, \ldots, x_n] = \exists y_1 \ldots \exists y_m \psi[x_1, \ldots, x_n, y_1, \ldots, y_m],$$

where ψ is a quantifier-free $\mathcal{L}$-formula. Given a structure $\mathcal{B}$ and elements $a_1, \ldots, a_n$ from a substructure $\mathcal{A}$ of $\mathcal{B}$, show that

$$\mathcal{A} \models \varphi[a_1, \ldots, a_n] \;\Rightarrow\; \mathcal{B} \models \varphi[a_1, \ldots, a_n].$$

b) Does the converse hold?

Exercise 2.

Let $\mathcal{L}$ be a language containing a unary predicate P_n for each n in $\mathbb{N}$. Consider an $\mathcal{L}$-theory T and an $\mathcal{L}$-formula $\varphi[x]$ such that in every model $\mathcal{A}$ of T every element a satisfying φ lies in one of the predicates $P_n^{\mathcal{A}}$. Show that there exists an N in $\mathbb{N}$ such that

$$T \models \forall x \left(\varphi[x] \to \bigvee_{n=0}^{N} P_n(x) \right).$$

Hint: THE theorem!

Exercise 3.

Given a (non-empty) set I, a *filter* $\mathcal{F}$ on I is a non-empty subset of the power set $\mathcal{P}(I)$ with the following properties:

1. $\emptyset \notin \mathcal{F}$ and $I \in \mathcal{F}$.
2. For all X and Y from $\mathcal{F}$ holds $X \cap Y$ in $\mathcal{F}$.
3. If X is in $\mathcal{F}$ and $X \subset Y$, then Y is also in $\mathcal{F}$.

a) Show that every intersection of filters is again a filter. Does the same hold for the union of filters?

A non-empty subset $\mathcal{B} \subseteq \mathcal{P}(I)$ is a *filter basis* if every finite intersection of elements from $\mathcal{B}$ is non-empty.

b) Show that ever filter basis $\mathcal{B}$ determines a filter which is generated by $\mathcal{B}$.

If $\mathcal{B} = \{X\}$ for a set $X \subset I$, then the filter generated by X is called *prinicipal filter*. An *ultrafilter* is a maximal filter (with respect to $\subset$).

(Please turn the page!)

c) Show that every filter is contained in an ultrafilter. Furthermore, show that a filter $\mathcal{F}$ is an ultrafilter if and only if it satisfies the following additional property:

4. If $X \cup Y$ is in $\mathcal{F}$, then X is in $\mathcal{F}$ or Y is in $\mathcal{F}$.

d) If a principal ultrafilter $\mathcal{U}$ is generated by a subset $X \subset I$, what is the size of X? For infinite I, let $\mathcal{F}(I)$ denote the collection of all cofinite subsets of I, that is,,

$$\mathcal{F}(I) = \{X \subset I : I \setminus X \text{ is finite}\}.$$

Show that $\mathcal{F}(I)$ is a filter. A ultrafilter $\mathcal{U}$ is non-principal if and only if $\mathcal{U}$ contains the filter $\mathcal{F}(I)$.

e) If the set I has cardinality $\aleph_0$, show that an ultrafilter $\mathcal{U}$ is closed under countable intersections if and only if $\mathcal{U}$ is a principal ultrafilter.

f) Given a filter $\mathcal{F}$ on I and a family $(X_i)_{i \in I}$ of (non-empty) sets, define the following relation on $\prod_{i \in I} X_i$:

$$(a_i)_{i \in I} \sim_{\mathcal{F}} (b_i)_{i \in I} \quad \Longleftrightarrow \quad \{i \in I : a_i = b_i\} \in \mathcal{F}.$$

Show that $\sim_{\mathcal{F}}$ is an equivalence relation.
